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The Galerkin method applied to convective instability problems

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The Galerkin method is applied in a new way to problems of stationary and oscillatory convective instability. By retaining the time derivatives in the equations rather than assuming an exponential time-dependence, the exact solution is approximated by the solution to a set of ordinary differential equations in time. Computations are simplified because the stability of this set of equations can be determined without finding the detailed solution. Furthermore, both stationary and oscillatory instability can be studied by means of the same trial functions. Previous studies which have treated only stationary instability by the Galerkin method can now be extended easily to include oscillatory instability. The method is illustrated for convective instability of a rotating fluid layer transferring heat.

1. Galerkin method

Convective instability problems determine the stability of a quiescent state characterized by $\mathbf{u} = 0$. The simplest example is that of fluid layer heated from below, which is treated by Chandrasekhar (1961) including the effects of rotation and a magnetic field. A similar problem concerns the stability of Couette flow between rotating cylinders, which is treated by Chandrasekhar (1961), Krueger & DiPrima (1964) and others. In both of these problems the study of oscillatory instability, in which the perturbation grows in an oscillatory manner, is inherently more difficult than is the study of stationary instability, in which the perturbation grows in an exponential manner. In the new approach, an initial value problem is solved approximately by the Galerkin method. The stability of the system is approximated by the solution to successively larger sets of ordinary differential equations, whose stability can be determined by several well-known techniques, including the Routh–Hurwitz criterion used here. Oscillatory instability can be studied along with stationary instability with little increase in effort.

For definiteness consider the equations governing the onset of convection in a fluid layer which is heated from below and rotated about a vertical axis. The basic equations, given by Chandrasekhar (1961, pp. 89–90), can be rearranged to the form:

$$\left.\begin{aligned}(D^2-a^2)\frac{\partial w}{\partial t} &= (D^2-a^2)^2 w - R^{\frac{1}{2}}a\theta - T^{\frac{1}{2}}Dz,\\ Pr\frac{\partial \theta}{\partial t} = (D^2-a^2)\,\theta + R^{\frac{1}{2}}aw, &\quad \frac{\partial z}{\partial t} = (D^2-a^2)\,z + T^{\frac{1}{2}}\,Dw,\end{aligned}\right\} \tag{1.1}$$

where w is the $\mathbf{e}_z$ component of velocity, θ is the temperature, z is the $\mathbf{e}_z$ component of vorticity, $R = g\alpha\beta d^4/\kappa\nu$ is the Rayleigh number, $T = 4\Omega^2 d^4/\nu^2$ is the Taylor number, $Pr = \nu/\kappa$ is the Prandtl number, g is the acceleration of gravity, α is the coefficient of thermal expansion, β is the magnitude of the initial temperature gradient, d is the thickness of the layer, κ is the thermal diffusivity, ν is the kinematic viscosity, Ω is the speed of rotation of the layer, a is the wavenumber, and $D \equiv \partial/\partial x$.

In the Galerkin method the solution is expanded in terms of a series with unknown coefficients which depend on time

$$\left.\begin{aligned} w(x,t) &= \sum_{i=1}^{M} A_i(t)\, W_i(x), \\ \theta(x,t) &= \sum_{i=1}^{M} B_i(t)\, \theta_i(x), \\ z(x,t) &= \sum_{i=1}^{M} C_i(t)\, Z_i(x). \end{aligned}\right\} \tag{1.2}$$

In the usual treatment (see DiPrima 1955, 1960; Krueger & DiPrima 1964) an exponential time-dependence is assumed, but by retaining the time derivatives we obtain computational and conceptual advantages which are enumerated below. The trial functions are substituted into (1.1) to form the residuals, which are made orthogonal (in the spatial domain) to each of the respective approximating functions. The resulting system of $3M$ ordinary differential equations can be represented

$$\bar{\bar{E}}\frac{d\bar{A}}{dt} = \bar{\bar{B}}\bar{A} \quad \text{or} \quad \frac{d\bar{A}}{dt} = \bar{\bar{E}}^{-1}\bar{\bar{B}}\bar{A} = \bar{\bar{D}}\bar{A} \tag{1.3}$$

where $\bar{A}^T = (A_1, B_1, C_1, \ldots, A_M, B_M, C_M)$ and B_{ki} and E_{ki} include terms of the form $[W_k, (D^2-a^2)\,W_i]$, etc., where $[u,v] \equiv \int_0^1 uv\,dx$ is the inner product. The inverse of the matrix E_{ki} exists if its determinant is non-zero, which is assumed. The stability of the system is governed by the stability of the set of ordinary equations (1.3) as a function of the parameters. The stability is determined for successively higher M, and if the results seem to converge the approximation is expected to represent the stability of the original system (1.1). The usual treatment by Galerkin's method can be recovered by setting $\bar{A} = e^{\sigma t}\bar{a}$:

$$(\bar{\bar{B}} - \sigma\bar{\bar{E}})\,\bar{a} = 0. \tag{1.4}$$

The chief advantage of viewing the system of ordinary differential equations rather than the equivalent set of algebraic equations is that powerful methods are available for studying the stability of (1.3) without actually solving it and these methods are equally applicable for stationary or oscillatory instability. Thus, authors who have previously applied the Galerkin method to determine only stationary instability can now easily study oscillatory instability simply by examining a different function of the parameters, as outlined below.

The solution to (1.3) is known (cf. Frazer, Duncan & Collar 1946, p. 288)

$$\bar{A} = \sum_{j=1}^{3M} \bar{C}_j e^{\lambda_j t} \tag{1.5}$$

where the $\bar{C}_j$ are constants if the eigenvalues λ_j are distinct. The system of equations is asymptotically stable if

$$\lim_{t\to\infty} \bar{A} = \bar{0} \tag{1.6}$$

and the necessary and sufficient condition that the system be asymptotically stable is that all the eigenvalues of D_{ij} have negative real parts (Gantmacher 1959, p. 144). The eigenvalues of D_{ij} can be calculated from

$$\det(D_{ij} - \lambda\delta_{ij}) = 0. \tag{1.7}$$

However, since

$$\det(B_{ij} - \lambda E_{ij}) = \det E_{ik} \det(D_{kj} - \lambda\delta_{kj}) \tag{1.8}$$

and E_{ik} is non-singular, the eigenvalues of D_{ij} are just the exponential time factor, σ, found from the usual Galerkin method, equation (1.4). The advantage of the new approach, using undetermined functions, $A_i(t)$, $B_i(t)$, $C_i(t)$, is that the eigenvalues of D_{ij} need not be calculated. Whether or not the eigenvalues have negative real parts can be decided by existing methods which are much simpler and shorter than the methods used to actually determine the eigenvalues. In particular, the Routh–Hurwitz criterion is especially useful.

The Routh–Hurwitz criterion gives necessary and sufficient conditions for all the roots of a real polynomial to have negative real parts. The determinantal equation (1.7) can be rewritten as the polynomial

$$\lambda^n + a_1\lambda^{n-1} + \dots + a_{n-2}\lambda^2 + a_{n-1}\lambda + a_n = 0. \tag{1.9}$$

The criterion is (cf. Gantmacher 1959, p. 231) that the roots all have negative real parts if

$$T_i > 0 \quad (i = 1, 2, \dots, n), \tag{1.10}$$

where T_i are the successive determinants formed from the matrix

$$\begin{bmatrix} a_1 & a_3 & a_5 & \dots & 0 \\ 1 & a_2 & a_4 & \dots & 0 \\ 0 & a_1 & a_3 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & a_n \end{bmatrix}, \tag{1.11}$$

i.e. $T_1 = a_1, T_2 = a_1a_2 - a_3$, etc. The coefficients a_i depend on the matrix D_{ij} (Aris 1962, p. 270)

$$a_i = (-1)^i \operatorname{tr}_i D, \tag{1.12}$$

where $\operatorname{tr}_p D$ is the sum of all the $p \times p$ determinants that can be formed with diagonal elements that are diagonal elements of D_{ij}. For example

$$\left.\begin{aligned} a_1 &= -\operatorname{tr} D, \\ a_n &= (-1)^n \det D. \end{aligned}\right\} \tag{1.13}$$

Additional information is given by Orlando's formula (Gantmacher 1959, p. 234): $T_{n-1} = 0$ if and only if the sum of at least one pair of roots of the polynomial is zero. Neutral oscillatory instability is characterized by $\sigma = \lambda = i\omega$. When the coefficients a_i are real, the complex roots of (1.9) can occur only in pairs, including the complex conjugates $\pm i\omega$. The sum of these two roots is zero, and $T_{n-1} = 0$ for neutral oscillatory instability. The case of neutral stationary instability is characterized by $\lambda = 0$, in which case $a_n = 0$.

The following procedure can be used to study the stability of the system of equations (1.3) as a function of Prandtl, Taylor and Rayleigh numbers. Duncan (1952, p. 117) first developed this procedure for the study of dynamical systems governed by sets of ordinary differential equations. For a specified value of Pr and T, consider the system as a function of R. The system is stable for $R = 0$ since then only dissipation mechanisms occur; all the Hurwitz determinants are positive. Increase R monotonically until a critical condition is reached such that any further change would result in instability. If one looks at the path of the eigenvalues in the complex plane as R is changed, the critical condition corresponds to an eigenvalue passing from the left half-plane, in which the real part is negative, to the right half-plane, in which the real part is positive. The eigenvalue can pass from one side to the other in two ways: either it will pass through zero, corresponding to neutral stationary instability and $a_n = 0$, or a pair of purely imaginary complex roots will exist, corresponding to neutral oscillatory instability and $T_{n-1} = 0$. Consequently, the stability of any system can be settled by examining a_n and T_{n-1} as functions of increasing R. Whichever condition

$$a_n = 0, \quad T_{n-1} = 0 \tag{1.14}$$

occurs first determines the type of instability, and the value of R there is the critical stability parameter. For that value of R the other Hurwitz determinants must be positive, of course.

The set of ordinary differential equations (1.3) is just an approximation to the system of partial differential equations (1.1). Successive approximations must be compared to ensure the approximation is a good one. For approximations beyond the first, the computations can conveniently be done on a computer, and the program code is suitable for many different stability problems.

It is clear that previous applications of the Galerkin method to stationary instability concentrated on satisfying the condition $a_n = 0$. Previous analyses can now easily be extended to include oscillatory instability simply by examining a different function of the parameters T_{n-1}. The Galerkin method, as outlined here, has computational advantages over the usual Galerkin method because it is much faster† to evaluate T_{n-1} than it is to find the roots to the polynomial (1.9) corresponding to the usual approach. Furthermore, certain features of the exact solution, such as the exchange of stabilities, are exemplified by the algebraic manipulations for the first approximation, as is illustrated below.

2. Example

The equations governing the onset of convection in a fluid layer which is heated from below and rotated about a vertical axis are given by (1.1) These equations are solved approximately here for the case of two rigid boundaries in

† It is eighty times faster using the standard subroutines available at the University of Minnesota for the Control Data 1604 computer.

order to illustrate the results that can be achieved for both stationary and oscillatory instability when using the same trial functions. The results are compared with the approximate results achieved by Chandrasekhar (1961), whose treatment required a great deal more computation. The method is applied to a new problem in the companion paper (Finlayson & Scriven 1968).

Since the basic results are already known, consider only the first approximation, which can be treated analytically. Three sets of approximating functions are assumed which satisfy the boundary conditions (see below) and each set is orthogonal. Equation (1.3) then simplifies to

$$\left.\begin{aligned}(\eta+a^2)\frac{dA}{dt} &= -(\alpha+\beta a^2+a^4)A+R^{\frac{1}{2}}a\delta B+T^{\frac{1}{2}}\epsilon C,\\ Pr\frac{dB}{dt} &= R^{\frac{1}{2}}a\delta A-(\nu+a^2)B,\\ \frac{dC}{dt} &= -T^{\frac{1}{2}}\epsilon A-(\zeta+a^2)C,\end{aligned}\right\} \tag{2.1}$$

where

$$\left.\begin{aligned}&\alpha = [D^4W, W], \quad \beta = -2[D^2W, W], \quad \delta = [\theta, W],\\ &\epsilon = [DZ, W] = -[Z, DW] \quad \text{when } W = 0 \text{ on the boundary},\\ &\zeta = -[D^2Z, Z], \quad \eta = -[D^2W, W], \quad \nu = -[D^2\theta, \theta].\end{aligned}\right\} \tag{2.2}$$

Equation (2.1) can be rearranged into the form of (1.3) where

$$\overline{\overline{E}} = \begin{bmatrix}\eta+a^2 & 0 & 0\\ 0 & Pr & 0\\ 0 & 0 & 1\end{bmatrix}, \quad \det E_{ij} \neq 0$$

and

$$\overline{\overline{B}} = \begin{bmatrix}-(\alpha+\beta a^2+a^4) & R^{\frac{1}{2}}a\delta & 0\\ R^{\frac{1}{2}}a\delta & -(\nu+a^2) & 0\\ 0 & 0 & -(\zeta+a^2)\end{bmatrix} + T^{\frac{1}{2}}\begin{bmatrix}0 & 0 & \epsilon\\ 0 & 0 & 0\\ -\epsilon & 0 & 0\end{bmatrix}. \tag{2.3}$$

The matrix B_{ij} has been split into its symmetric and antisymmetric parts. When the Taylor number is zero, the matrix is symmetric and can have only real eigenvalues; hence oscillatory instability is impossible. This is a proof of the principle of exchange of stabilities, shown for the non-rotating layer by Pellew & Southwell (1940). It is clearly evident here because of the way the equations were non-dimensionalized, following the procedure first suggested by Sani (1963). The matrix B must have a certain amount of asymmetry before oscillatory instability occurs. The matrix D in (1.3) becomes

$$\overline{\overline{D}} = \begin{bmatrix}-C/B & R^{\frac{1}{2}}E/B & T^{\frac{1}{2}}D/B\\ R^{\frac{1}{2}}E/Pr & -A/Pr & 0\\ -T^{\frac{1}{2}}D & 0 & -F\end{bmatrix}, \tag{2.4}$$

where $A = \nu+a^2$, $B = \eta+a^2$, $C = \alpha+\beta a^2+a^4$, $D = \epsilon$, $E = a\delta$ and $F = \zeta+a^2$.

The stability of this system is determined by $\det D_{ij} = 0$ for stationary instability (1.14) or $T_2 = a_1 a_2 - a_3 = 0$ for oscillatory instability, where $a_1 = -\operatorname{tr} D$, $a_2 = \operatorname{tr}_2 D$, and $a_3 = -\det D$. The results are

$$\left.\begin{aligned} R_c^s &= \frac{CA + TD^2A/F}{E^2}, \\ R_c^o &= \frac{A+FPr}{E^2}\left[C+BF+TD^2Pr^2\,\frac{C+FB}{(CPr+AB)(A+FPr)}\right], \end{aligned}\right\} \tag{2.5}$$

and when oscillatory instability occurs the frequency is given by

$$\omega^2 = TD^2\,\frac{A-FPr}{AB+CPr} - F^2. \tag{2.6}$$

To this point in the analysis all results apply to the exact solution, provided we interpret the functions W, θ and Z in equations (2.2) as the exact solution. The validity of the principle of exchange of stabilities is clearly evident, as is the dependence of R_c and ω on the parameters of the problem. For the approximate solution these expressions must be minimized with respect to the wave-number a in order to find the lowest possible Rayleigh number for any disturbance. This computation is most conveniently done on a computer. Note that while the numerical values in (2.5), (2.6) depend upon the trial solution (and hence the boundary conditions) the structure of the equations does not. Thus the same computer program can calculate results for different boundary conditions, corresponding to the different numerical values of the integrals.

The results for very large Taylor number are

$$\left.\begin{aligned} R_c^s &\to \tfrac{3}{2}2^{\frac{1}{3}}T^{\frac{2}{3}}\epsilon^{\frac{4}{3}}\delta^{-2}, \\ a_c &\to 2^{-\frac{1}{6}}T^{\frac{1}{6}}\epsilon^{\frac{1}{3}}, \end{aligned}\right\} \quad T\to\infty, \tag{2.7}$$

while for oscillatory instability

$$\left.\begin{aligned} R_c^o &\to 6(1+Pr)\left[\frac{Pr^2T}{2(1+Pr)^2}\right]^{\frac{2}{3}}\epsilon^{\frac{4}{3}}\delta^{-2}, \\ a_c &\to (\tfrac{1}{2}T)^{\frac{1}{6}}\left[\frac{Pr}{1+Pr}\right]^{\frac{1}{3}}\epsilon^{\frac{1}{3}}, \\ \omega &\to \frac{(2-3Pr^2)^{\frac{1}{2}}}{[Pr(1+Pr)]^{\frac{1}{3}}}\left(\frac{T}{2}\right)^{\frac{1}{3}}\epsilon^{\frac{2}{3}}, \end{aligned}\right\} \quad \text{as} \quad Pr^2T\to\infty. \tag{2.8}$$

The type of instability which occurs will have the lower value of the critical Rayleigh number. In the limit of large Taylor number, the critical Rayleigh number for stationary instability is always less than that for oscillatory instability whenever the Prandtl number is greater than 0·67659. This result is derived in the same manner used by Chandrasekhar (1961, p. 118) to prove the result for the exact solution for free boundaries. However, the new result for rigid–rigid or rigid–free boundaries applies only to the first approximation.

In order to obtain numerical results, the approximating functions are chosen to be orthonormal functions satisfying the boundary conditions, which for rigid boundaries are

$$W = DW = \theta = Z = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{2.9}$$

Derived, or secondary, boundary conditions are obtained by requiring that the differential equation (1.1) be satisfied on the boundary:

$$D^2\theta = D^2Z = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{2.10}$$

The W and θ functions are even while Z is an odd function. A set of approximating functions which satisfies these requirements is

$$\left.\begin{aligned} W(x) &= C_1(x) = \frac{\cosh \lambda_1 x}{\cosh \frac{1}{2}\lambda_1} - \frac{\cos \lambda_1 x}{\cos \frac{1}{2}\lambda_1}, \\ \theta(x) &= 2^{\frac{1}{2}} \cos \pi x, \\ Z(x) &= 2^{\frac{1}{2}} \sin 2\pi x. \end{aligned}\right\} \tag{2.11}$$

Integral	Fixed–fixed boundaries
α	500·5639
β	24·65216
δ	0·9862404
ϵ	3·493051
ζ	39·47842
η	12·32608
ν	9·869604

TABLE 1. Values of integrals

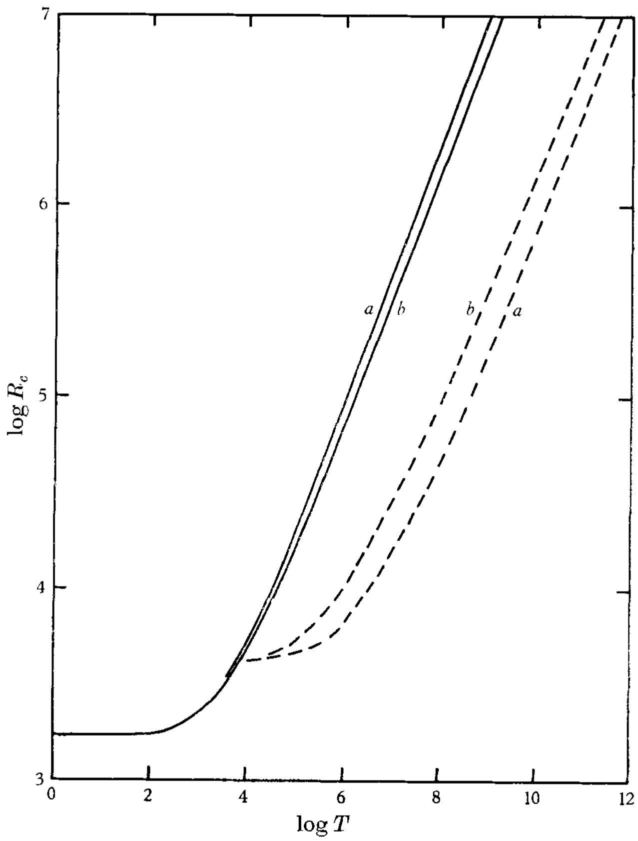

FIGURE 1. The variation of R_c with T. $Pr = 0{\cdot}025$, ——, stationary instability; ––––, oscillatory instability. (a) this work; (b) Chandrasekhar (1961).

These functions give rise to the values of the integrals listed in table 1, and the velocity functions, $C_1(x)$, are derived by Reid & Harris (1958).

Illustrative results are presented in figure 1. While not as accurate as Chandrasekhar's approximate solution, the results are reasonable as a first approximation considering the greater ease with which they were determined. Higher approximations have not been calculated since most of the results are already known. The accuracy of the asymptotic formula (2.7) can be assessed by comparing $R \rightarrow 10{\cdot}0T^{\frac{2}{3}}$ obtained there as a first approximation, to $R \rightarrow 8{\cdot}69T^{\frac{2}{3}}$ obtained by Niiler & Bisshopp (1965) in a rigorous asymptotic analysis. For oscillatory instability, the only computations that exist are for $Pr = 0{\cdot}025$, and equations (2.8) give answers which differ from Chandrasekhar's by 41% in R, 6% in a and 22% in ω for $T = 10^{12}$. Equations (2.8) can then be used for other Prandtl numbers as a first approximation with about this degree of accuracy. Note also that (2.7) and (2.8) are valid for the case of rigid–free boundaries; only the values of ϵ and δ differ corresponding to different trial functions for W and Z.

3. Conclusion

By retaining the time derivatives in the equations and using the Galerkin method to reduce the set of partial differential equations to a set of ordinary differential equations, it is possible to study easily both stationary and oscillatory instability. Previous analyses which have used the Galerkin method to study stationary instability can be extended to include oscillatory instability simply by looking at another function of the parameters, T_{n-1}. This application of the Galerkin method provides a powerful tool for finding approximate solutions to convective instability problems.

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